

# ASYMPTOTICS OF GENERALIZED GALOIS NUMBERS VIA AFFINE KAC-MOODY ALGEBRAS

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**ABSTRACT.** Generalized Galois numbers count the number of flags in vector spaces over finite fields. Asymptotically, as the dimension of the vector space becomes large, we give their exponential growth and determine their initial values. The initial values are expressed analytically in terms of theta functions and Euler's generating function for the partition numbers. Our asymptotic enumeration method is based on a Demazure module limit construction for integrable highest weight representations of affine Kac-Moody algebras. For the classical Galois numbers, that count the number of subspaces in vector spaces over finite fields, the theta functions are Jacobi theta functions. We apply our findings to the asymptotic number of linear  $q$ -ary codes, and conclude with some final remarks about possible future research concerning asymptotic enumerations via limit constructions for affine Kac-Moody algebras and modularity of characters of integrable highest weight representations.

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## 1. INTRODUCTION

The generalized Galois numbers  $G_N^{(r)}(q)$  count the number of flags  $0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_r = \mathbf{F}_q^N$  of length  $r$  in an  $N$ -dimensional vector space over a field with  $q$  elements [26]. In particular, when  $r = 2$  these are the classical Galois numbers studied by Goldman and Rota [7] which give the total number of subspaces in  $\mathbf{F}_q^N$ .

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We show that the generalized Galois numbers grow asymptotically, as  $r$  is fixed and  $N \rightarrow \infty$ , exponentially with factor  $O(N^2)$  in logarithmic “time” scale:

$$G_N^{(r)}(q) \sim I_r(q) \cdot e^{O(N^2) \log(q)}.$$

Here, “time” equals the cardinality of the finite field. Our main result is the explicit description of the initial values  $I_r(q)$  via theta functions and Euler’s generating function for the partition numbers.

This investigation serves three purposes. First, the generalized Galois numbers are of independent interest as they enumerate points in fundamental geometric objects defined over finite fields. For example, by definition the classical Galois numbers

$$G_N(q) = G_N^{(2)}(q) = \sum_{k=0}^N |\mathrm{Gr}(k, N)(\mathbf{F}_q)|$$

count the number of  $\mathbf{F}_q$ -rational points in Grassmann varieties. The numbers of solutions of the set of equations for  $\mathrm{Gr}(k, N)$  in extension fields  $\mathbf{F}_{p^n}$  of  $\mathbf{F}_p$  are in turn subject to the study of local zeta-functions  $Z(\mathrm{Gr}(k, N), t) = \exp(\sum_{n \geq 1} |\mathrm{Gr}(k, N)(\mathbf{F}_{p^n})| \frac{t^n}{n})$  in number theory. Let us mention that a generating function for the local zeta-function  $Z(\mathrm{Gr}(k, N), t)$  can be given by

$$Z(\mathrm{Gr}(k, N), t) = \frac{1}{(1-t)^{b_0} (1-pt)^{b_1} \dots (1-p^{k(N-k)}t)^{b_{k(N-k)}}},$$

where the  $b_i = \dim H_{2i}(\mathrm{Gr}(k, N)(\mathbf{C}), \mathbf{Z})$  are the even topological Betti numbers of the complex Grassmannian. Consequently, the study of Galois numbers reflects upon many subjects.

Second, the Galois numbers enumerate asymptotically the number of equivalence classes of linear  $q$ -ary codes in algebraic coding theory as recently shown by Hou and Wild [9, 10, 11, 28, 29]. For example, the asymptotic number  $N_{n,q}^{\mathfrak{S}}$  of linear  $q$ -ary codes under permutation equivalence is

$$N_{n,q}^{\mathfrak{S}} \sim \frac{G_n(q)}{n!}.$$

We apply our findings to those asymptotic equivalences, and derive considerable simplifications of the asymptotic enumeration of linear  $q$ -ary codes (§4).

Third, our investigation serves the demonstration of the asymptotic enumeration method itself (§3). We identify the generalized Galois numbers  $G_N^{(r)}(q)$  as the basic specialization of the Demazure modules  $V_{-N\omega_1}(\Lambda_0)$  of the affine Kac-Moody algebra  $\widehat{\mathfrak{sl}}_r$  (see (3.14)). Those characters pass via a graded limit construction [6, 17, 20, 21] to the characters of the fundamental representations of our affine Kac-Moody algebra:

$$\lim_{n \rightarrow \infty} \chi(V_{-(rn+j)\omega_1}(\Lambda_0)) = \chi(V(\Lambda_j)).$$

By a symmetry argument, Kac's [12] character formula for the basic representation

$$\chi(V(\Lambda_0)) = \sum_{k=0}^{\infty} p^{(r-1)}(k) e^{\Lambda_0 - k\delta} \cdot \sum_{\gamma \in Q} e^{-(h\|\gamma\|^2\delta + \gamma)}$$

then allows us to prove our main result:

**Theorem 3.2.** *Consider the generalized Galois number  $G_N^{(r)}(q)$ . For any prime power  $e^\delta = p^m$  (in fact for any complex number  $e^\delta$  where  $\delta \in -2\pi i\mathbf{H}$ ) and  $0 \leq j < r$  we have the limit*

$$\lim_{n \rightarrow \infty} G_{rn+j}^{(r)}(e^\delta) \cdot e^{-u_j(r,n)\delta} = \frac{\Theta_{F_j}(-\frac{\delta}{2\pi i})}{\phi(e^{-\delta})^{r-1}}.$$

Here,  $\phi(x)^{-1} = \prod_{m=1}^{\infty} (1 - x^m)^{-1}$  denotes Euler's generating function for the partition numbers, and  $\Theta_{F_j}(z) = \sum_{\mathbf{k} \in \mathbf{Z}^{r-1}} e^{2\pi i z F_j(\mathbf{k})}$  are theta functions associated to the quadratic forms  $F_0, F_1, \dots, F_{r-1}$  on the lattice  $\mathbf{Z}^{r-1}$  given by

$$F_0(k_1, \dots, k_{r-1}) = \sum_{l=1}^{r-1} k_l^2 - \sum_{l=1}^{r-2} k_l k_{l+1},$$

$$F_j(k_1, \dots, k_{r-1}) = \left(k_j + \frac{1}{2}\right)^2 + \sum_{l=1, l \neq j}^{r-1} k_l^2 - \sum_{l=1}^{r-2} k_l k_{l+1}.$$

The exponents  $u_0, u_1, \dots, u_{r-1}$  are

$$u_0(r, n) = \frac{r(r-1)n^2}{2},$$

$$u_j(r, n) = \frac{(rn+j)(rn+j-1)}{2} - \frac{rn(rn+2j-r)}{2r} + \frac{1}{4}.$$

For the classical Galois numbers our theta functions turn out to be Jacobi theta functions (see Corollary 3.6).

Let us conclude the introduction with the following remark on our asymptotic enumeration method. In the case of generalized Galois numbers we do not make use of the modularity of characters of integrable highest weight modules, since the prime powers  $p^{-m} < 1$  lie in the region of convergence of our modular forms. However, we will discuss, in §5, an important eventual application of our asymptotic enumeration method where modularity has to be exploited.

## 2. NOTATION AND BACKGROUND

The generalized Galois number  $G_N^{(r)}(q) \in \mathbf{N}[q]$  can be defined as the specialization of the generalized  $N$ -th Rogers-Szegő polynomial at  $(\mathbf{1}, q)$  [26]:

$$G_N^{(r)}(q) = H_N^{(r)}(\mathbf{1}, q).$$

The  $N$ -th generalized Rogers-Szegő polynomial  $H_N^{(r)}(\mathbf{z}, q) \in \mathbf{N}[z_1, \dots, z_r, q]$  [22, 23, 25] (see [1] for an account) is defined as the generating function of the  $q$ -multinomial coefficients:

$$H_N^{(r)}(\mathbf{z}, q) = \sum_{\substack{\mathbf{k}=(k_1, \dots, k_r) \in \mathbf{N}^r \\ k_1 + \dots + k_r = N}} \begin{bmatrix} N \\ \mathbf{k} \end{bmatrix}_q \mathbf{z}^{\mathbf{k}}.$$

Recall from [26] that the  $q$ -multinomial coefficient  $\begin{bmatrix} N \\ k_1, \dots, k_r \end{bmatrix}_q$  counts the number of flags  $0 = V_0 \subseteq \dots \subseteq V_r = \mathbf{F}_q^N$  subject to the conditions  $\dim(V_i) = k_1 + \dots + k_i$ .

For general facts about affine Kac-Moody algebras and their representation theory we refer the reader to [3, 13], and for Demazure modules to [6]. Let us briefly fix the notation we will use throughout. We consider the affine Kac-Moody algebra  $\widehat{\mathfrak{sl}}_r$ . We denote the simple roots by  $\alpha_0, \alpha_1, \dots, \alpha_{r-1}$ , the highest root by  $\theta = \alpha_1 + \dots + \alpha_{r-1}$  and the imaginary root by  $\delta = \alpha_0 + \theta$ . The affine root lattice is then defined as  $\widehat{Q} = \mathbf{Z}\alpha_0 \oplus \mathbf{Z}\alpha_1 \oplus \dots \oplus \mathbf{Z}\alpha_{r-1}$  and the real span of the simple roots is given by  $\widehat{\mathfrak{h}}_{\mathbf{R}}^* = \mathbf{R} \otimes_{\mathbf{Z}} \widehat{Q}$ . We have a non-degenerate symmetric bilinear form on  $\widehat{\mathfrak{h}}_{\mathbf{R}}^*$  by  $\langle \alpha_i, \alpha_j \rangle = c_{ij}$  where  $C = (c_{ij})$  is the Cartan matrix of  $\widehat{\mathfrak{sl}}_r$ , and define  $\|\cdot\|^2 = (2h)^{-1} \langle \cdot, \cdot \rangle$  where  $h = r$  is the Coxeter number of  $\widehat{\mathfrak{sl}}_r$ . For a dominant integral weight  $\Lambda = m_1\Lambda_0 + m_2\Lambda_1 + \dots + m_{r-1}\Lambda_{r-1}$  we let  $V(\Lambda)$  be the integrable highest weight representation of weight  $\Lambda$  of  $\widehat{\mathfrak{sl}}_r$  and  $\chi(V(\Lambda))$  its character. The  $\Lambda_0, \Lambda_1, \dots, \Lambda_{r-1}$  are called fundamental weights, the  $V(\Lambda_l)$  the fundamental representations and  $V(\Lambda_0)$  the basic representation. As for the Demazure modules, we will only consider the translations  $t_{-k\omega_1} = (s_1 s_2 \dots s_{r-1} \sigma^{r-1})^k$  in the extended affine Weyl group of  $\widehat{\mathfrak{sl}}_r$ , where  $\omega_1 = \Lambda_1 - \Lambda_0$ . Here,  $\sigma$  denotes the automorphism of the Dynkin diagram of  $\widehat{\mathfrak{sl}}_r$  which sends 0 to 1, and  $s_1, \dots, s_{r-1}$  are the simple reflections associated to the simple roots  $\alpha_1, \dots, \alpha_{r-1}$ . We denote the Demazure module associated to those translations by  $V_{-k\omega_1}(\Lambda)$  and its character by  $\chi(V_{-k\omega_1}(\Lambda))$ . We write the monomials in the characters of our modules as  $e^\lambda$ , the coefficient  $k$  in the monomial  $e^{-k\alpha_0}$  is referred to as the degree.

$\mathbf{H}$  will denote the upper half plane in  $\mathbf{C}$ . We write  $\sim$  for asymptotic equivalence, that is for  $f, g : \mathbf{N} \rightarrow \mathbf{R}_{>0}$  we write  $f(n) \sim g(n)$  if  $\lim_{n \rightarrow \infty} f(n)/g(n) = 1$ .

### 3. ASYMPTOTICS OF GENERALIZED GALOIS NUMBERS

Let us start with a direct consequence of Kac's character formula [12, (3.37)].

**Proposition 3.1.** *Consider the basic representation  $V(\Lambda_0)$  of  $\widehat{\mathfrak{sl}}_r$ . Let  $Q$  be the lattice  $Q = \widehat{Q}/\mathbf{Z}\alpha_0 = \mathbf{Z}\alpha_1 \oplus \dots \oplus \mathbf{Z}\alpha_{r-1} \cong \mathbf{Z}^{r-1}$ . Then,*

$$(3.1) \quad \chi(V(\Lambda_0)) = \frac{e^{\Lambda_0}}{\phi(e^{-\delta})^{r-1}} \cdot \sum_{\mathbf{k} \in \mathbf{Z}^{r-1}} e^{\frac{1}{4}\theta} \prod_{l=0}^{r-1} e^{-F_l(\mathbf{k})\alpha_l}.$$

Here,  $\phi(x)^{-1} = \prod_{i=1}^m (1 - x^m)^{-1}$  is Euler's generating function for the partition numbers, and the  $F_0, F_1, \dots, F_{r-1}$  are quadratic forms on the lattice  $Q \cong \mathbf{Z}^{r-1}$  defined as

$$(3.2) \quad F_0(k_1, \dots, k_{r-1}) = \sum_{l=1}^{r-1} k_l^2 - \sum_{l=1}^{r-2} k_l k_{l+1},$$

$$(3.3) \quad F_j(k_1, \dots, k_{r-1}) = \left(k_j + \frac{1}{2}\right)^2 + \sum_{l=1, l \neq j}^{r-1} k_l^2 - \sum_{l=1}^{r-2} k_l k_{l+1}.$$

*Proof.* Due to Kac [12, (3.37)] we have the following character formula for the basic representation  $V(\Lambda_0)$ :

$$(3.4) \quad \chi(V(\Lambda_0)) = \sum_{k=0}^{\infty} p^{(r-1)}(k) e^{\Lambda_0 - k\delta} \cdot \sum_{\gamma \in Q} e^{-(h\|\gamma\|^2\delta + \gamma)}$$

The function  $p^{(r-1)}(k)$  is defined via its generating function  $\sum_{k=0}^{\infty} p^{(r-1)}(k) x^k = \phi(x)^{-r+1}$ . Let  $f(\mathbf{k})$  be the quadratic form  $f(\mathbf{k}) = f(k_1, \dots, k_{r-1}) = \sum_{i=1}^{r-1} k_i^2 - \sum_{i=1}^{r-2} k_i k_{i+1}$  on the lattice  $Q \cong \mathbf{Z}^{r-1}$ . If we express  $\gamma \in Q$  as the linear combination  $\gamma = k_1\alpha_1 + \dots + k_{r-1}\alpha_{r-1}$ , we have  $h\|\gamma\|^2 = f(\mathbf{k})$ . Then,

$$\begin{aligned} \chi(V(\Lambda_0)) &= \sum_{k=0}^{\infty} p^{(r-1)}(k) e^{\Lambda_0 - k\delta} \cdot \sum_{\gamma \in Q} e^{-(h\|\gamma\|^2\delta + \gamma)} \\ &= \frac{e^{\Lambda_0}}{\phi(e^{-\delta})^{r-1}} \cdot \sum_{\gamma \in Q} e^{-(h\|\gamma\|^2\delta + \gamma)} \\ &= \frac{e^{\Lambda_0}}{\phi(e^{-\delta})^{r-1}} \cdot \sum_{\mathbf{k} \in \mathbf{Z}^{r-1}} e^{-f(\mathbf{k})\delta} e^{-k_1\alpha_1} \dots e^{-k_{r-1}\alpha_{r-1}} \\ &= \frac{e^{\Lambda_0}}{\phi(e^{-\delta})^{r-1}} \cdot \sum_{\mathbf{k} \in \mathbf{Z}^{r-1}} e^{-f(\mathbf{k})\alpha_0} e^{-(f(\mathbf{k})+k_1)\alpha_1} \dots e^{-(f(\mathbf{k})+k_{r-1})\alpha_{r-1}}. \end{aligned}$$

Note that  $F_0(\mathbf{k}) = f(\mathbf{k})$  and  $F_j(\mathbf{k}) - \frac{1}{4} = f(\mathbf{k}) + k_j$ . This finishes the proof.  $\square$

We are ready to prove our main result.

**Theorem 3.2.** *Consider the generalized Galois number  $G_N^{(r)}(q)$ . For any prime power  $e^\delta = p^m$  (in fact for any complex number  $e^\delta$  where  $\delta \in -2\pi i\mathbf{H}$ )*

and  $0 \leq j < r$  we have the limit

$$(3.5) \quad \lim_{n \rightarrow \infty} G_{rn+j}^{(r)}(e^\delta) \cdot e^{-u_j(r,n)\delta} = \frac{\Theta_{F_j}(-\frac{\delta}{2\pi i})}{\phi(e^{-\delta})^{r-1}}.$$

Here,  $\phi(x)^{-1} = \prod_{m=1}^{\infty} (1 - x^m)^{-1}$  denotes Euler's generating function for the partition numbers, and  $\Theta_{F_j}(z) = \sum_{\mathbf{k} \in \mathbf{Z}^{r-1}} e^{2\pi i z F_j(\mathbf{k})}$  are theta functions associated to the quadratic forms  $F_0, F_1, \dots, F_{r-1}$  on the lattice  $\mathbf{Z}^{r-1}$  given by

$$(3.6) \quad F_0(k_1, \dots, k_{r-1}) = \sum_{l=1}^{r-1} k_l^2 - \sum_{l=1}^{r-2} k_l k_{l+1},$$

$$(3.7) \quad F_j(k_1, \dots, k_{r-1}) = \left(k_j + \frac{1}{2}\right)^2 + \sum_{l=1, l \neq j}^{r-1} k_l^2 - \sum_{l=1}^{r-2} k_l k_{l+1}.$$

The exponents  $u_0, u_1, \dots, u_{r-1}$  are

$$(3.8) \quad u_0(r, n) = \frac{r(r-1)n^2}{2},$$

$$(3.9) \quad u_j(r, n) = \frac{(rn+j)(rn+j-1)}{2} - \frac{rn(rn+2j-r)}{2r} + \frac{1}{4}.$$

*Proof.* Let  $N = rn + j$  for  $0 \leq j < r$  and consider the Demazure module  $V_{-N\omega_1}(\Lambda_0)$  associated to the translation  $t_{-N\omega_1} = (s_1 s_2 \dots s_{r-1} \sigma^{r-1})^N$ . By Sanderson [24] we can describe its character  $\chi(V_{-N\omega_1}(\Lambda_0))$  via a certain specialization of a symmetric Macdonald polynomial (see [19, Chapter VI] for their definition and properties). That is, let  $[N] = (N, 0, \dots, 0) \in \mathbf{N}^r$  denote the one-row Young diagram, and  $\eta_N$  the smallest composition of degree  $N$ , i.e. since  $N = rn + j$  we have  $\eta_N = ((n)^{r-j}, (n+1)^j) \in \mathbf{N}^r$ . Following [24, §2] we have  $[N] = t_{-N\omega_1} \cdot \eta_N$  with the convention  $\sigma \cdot \eta_N = \eta_N$ . Furthermore, by computing the expression  $u([N]) - u(\eta_N)$  in [24, Theorem 6], the maximal occurring degree in  $\chi(V_{-N\omega_1}(\Lambda_0))$  is given by

$$(3.10) \quad d_r(N) = d_r(rn + j) = \frac{(rn+j)(rn+j-1)}{2} - \frac{rn(rn+2j-r)}{2r}.$$

Note that  $d_r(rn) = u_0(r, n)$  and  $d_r(rn+j) = u_j(r, n) - \frac{1}{4}$  for  $j = 1, \dots, r-1$ . Let  $\mathbf{z} = (e^{\Lambda_1 - \Lambda_0}, e^{\Lambda_2 - \Lambda_1}, \dots, e^{\Lambda_{r-1} - \Lambda_{r-2}}, e^{\Lambda_0 - \Lambda_{r-1}})$ . Then, by [24, Theorem 6 and 7]<sup>1</sup> we have

$$(3.11) \quad \chi(V_{-N\omega_1}(\Lambda_0)) = e^{\Lambda_0 - d_r(N)\delta} \cdot P_{[N]}(\mathbf{z}; e^\delta, 0),$$

where  $P_{[N]}(\mathbf{z}; q, 0)$  denotes the specialized symmetric Macdonald polynomial associated to the partition  $[N]$ . Furthermore, by Hikami [8, Equation

<sup>1</sup>There seems to be a missprint in [24, §4]. Namely, the image  $\pi(q)$  should equal  $q = e^\delta$ , not  $q = e^{-\delta}$ .

(3.4)] this Macdonald polynomial equals the  $N$ -th generalized Rogers-Szegő polynomial:

$$(3.12) \quad P_{[N]}(\mathbf{z}; q, 0) = H_N^{(r)}(\mathbf{z}, q).$$

Combining (3.11) and (3.12) we obtain

$$(3.13) \quad \chi(V_{-N\omega_1}(\Lambda_0)) = e^{\Lambda_0 - d_r(N)\delta} \cdot H_N^{(r)}(\mathbf{z}, e^\delta).$$

Consequently, the basic specialization at  $e^{-\alpha_1} = \dots = e^{-\alpha_{r-1}} = 1$  of the Demazure character on the left-hand side of (3.13) gives the generalized Galois number  $G_N^{(r)}$  up to translation:

$$(3.14) \quad \chi(V_{-N\omega_1}(\Lambda_0))|_{(e^{-\alpha_1}=\dots=e^{-\alpha_{r-1}}=1)} = e^{\Lambda_0 - d_r(N)\delta} \cdot G_N^{(r)}(e^\delta).$$

Now, let us proceed to the limit considerations. By Fourier and Littelmann [6, Theorem D] (which is based on work by Mathieu and Kumar [17, 20, 21]) the characters of our Demazure modules pass, as  $N \rightarrow \infty$ , as functions in  $(e^{-\alpha_0}, e^{-\alpha_1}, \dots, e^{-\alpha_{r-1}})$  to the characters of the fundamental representations  $V(\Lambda_0), V(\Lambda_1), \dots, V(\Lambda_{r-1})$  of  $\widehat{\mathfrak{sl}}_r$  as follows

$$(3.15) \quad \lim_{n \rightarrow \infty} \chi(V_{-(rn+j)\omega_1}(\Lambda_0)) = \chi(V(\Lambda_j)).$$

We are ready to prove our claimed identity (3.5) in the case  $j = 0$ . Recall that  $\delta = \alpha_0 + \theta$  where  $\theta = \alpha_1 + \dots + \alpha_{r-1}$ , and  $d_r(rn) = u_0(r, n)$ . Then, the equations (3.14), (3.15) and the character formula (3.1) for the basic representation  $V(\Lambda_0)$  imply

$$\begin{aligned} \lim_{n \rightarrow \infty} G_{rn}^{(r)}(e^\delta) \cdot e^{-u_0(r, n)\delta} &= e^{-\Lambda_0} \cdot \chi(V(\Lambda_0))|_{(e^{-\alpha_1}=\dots=e^{-\alpha_{r-1}}=1)} \\ &= \frac{1}{\phi(e^{-\delta})^{r-1}} \cdot \sum_{\mathbf{k} \in \mathbf{Z}^{r-1}} e^{-F_0(\mathbf{k})\delta} \\ &= \frac{1}{\phi(e^{-\delta})^{r-1}} \cdot \Theta_{F_0} \left( -\frac{\delta}{2\pi i} \right). \end{aligned}$$

There is a subtlety to our deduction in the cases  $j = 1, \dots, r-1$ . The characters of the representations  $V(\Lambda_0), V(\Lambda_1), \dots, V(\Lambda_{r-1})$  are symmetrical in the sense that they subsequently differ by an application of the automorphism  $\sigma$  that sends 0 to 1 in the Dynkin diagram of  $\widehat{\mathfrak{sl}}_r$ . An application of  $\sigma$  cyclically shifts the fundamental weights  $\Lambda_i \mapsto \Lambda_{i+1}$  and the simple roots  $\alpha_i \mapsto \alpha_{i+1}$  (by cyclic we mean  $\Lambda_r = \Lambda_0$  and  $\alpha_r = \alpha_0$ ). Consequently, it leaves  $\delta$  invariant  $\sigma(\delta) = \delta$  and  $\sigma^j(\theta) = \alpha_0 + \theta - \alpha_j$  for  $j = 1, \dots, r-1$ . To be precise, Kac's character formula (3.1) for the fundamental representations  $V(\Lambda_1), \dots, V(\Lambda_{r-1})$  reads as follows

$$(3.16) \quad \chi(V(\Lambda_j)) = \frac{e^{\Lambda_j}}{\phi(e^{-\delta})^{r-1}} \cdot \sum_{\mathbf{k} \in \mathbf{Z}^{r-1}} e^{\frac{1}{4}(\alpha_0 + \theta - \alpha_j)} \prod_{l=0}^{r-1} e^{-F_{l+r-j}(\mathbf{k})\alpha_l}.$$

Recall that  $u_j(r, n) = d_r(rn + j) + \frac{1}{4}$ . Therefore, we obtain

$$\begin{aligned}
\lim_{n \rightarrow \infty} G_{rn+j}^{(r)}(e^\delta) \cdot e^{-u_j(r, n)\delta} &= e^{-\frac{1}{4}\delta} \cdot \lim_{n \rightarrow \infty} G_{rn+j}^{(r)}(e^\delta) \cdot e^{-d_r(rn+j)\delta} \\
&= e^{-\frac{1}{4}\delta} \cdot e^{-\Lambda_j} \cdot \chi(V(\Lambda_j))|_{(e^{-\alpha_1}=\dots=e^{-\alpha_{r-1}}=1)} \\
&= \frac{e^{-\frac{1}{4}\delta}}{\phi(e^{-\delta})^{r-1}} \cdot \sum_{\mathbf{k} \in \mathbf{Z}^{r-1}} e^{\frac{1}{4}\delta} e^{-F_j(\mathbf{k})\delta} \\
&= \frac{1}{\phi(e^{-\delta})^{r-1}} \cdot \Theta_{F_j} \left( -\frac{\delta}{2\pi i} \right).
\end{aligned}$$

This establishes the theorem.  $\square$

**Remark 3.3.** Motivated by Kac's character formula (3.4) we write the quadratic forms on  $Q = \mathbf{Z}\alpha_1 \oplus \dots \oplus \mathbf{Z}\alpha_{r-1} \cong \mathbf{Z}^r$  intrinsically in terms of data associated to our affine Kac-Moody algebra  $\widehat{\mathfrak{sl}}_r$  as follows. For  $j = 1, \dots, r-1$  one has

$$(3.17) \quad F_0(\gamma) = h||\gamma||^2$$

$$(3.18) \quad F_j(\gamma) = h||\gamma||^2 + \langle \Lambda_j, \gamma \rangle + \frac{1}{4}.$$

The exponents  $u_0, u_1, \dots, u_{r-1}$  can be described via the translation formula [13, (6.5.3)].

**Remark 3.4.** One can phrase Theorem 3.2 asymptotically as

$$(3.19) \quad G_{rn+j}^{(r)}(e^\delta) \sim \frac{\Theta_{F_j}(-\frac{\delta}{2\pi i})}{\phi(e^{-\delta})^{r-1}} \cdot (e^\delta)^{u_j(n, r)}$$

Note that for fixed  $r$  the exponents  $u_j(n, r)$  lie in  $O(n^2)$ .

**Remark 3.5.** For  $j = 1, \dots, r-1$  the limits in Theorem 3.2 coincide. In fact, the quadratic forms  $F_1, \dots, F_{r-1}$  differ only by a cyclic shift of the coordinates. Summation over the complete lattice  $\mathbf{Z}^{r-1}$  produces equality.

Let us summarize the implications of Theorem 3.2 for the classical Galois numbers  $G_N(q) = G_N^{(2)}(q)$  that count the number of subspaces in  $\mathbf{F}_q^N$ .

**Corollary 3.6.** *Consider the classical Galois numbers  $G_N(q)$ . For any prime power  $q = p^m$  (in fact for any complex number  $|q| > 1$ ) we have*

$$(3.20) \quad g_{2\infty+1}(q) = \lim_{n \rightarrow \infty} G_{2n+1}(q) \cdot q^{-\frac{(2n+1)^2}{4}} = \frac{\vartheta_2(0, q^{-1})}{\phi(q^{-1})},$$

$$(3.21) \quad g_{2\infty}(q) = \lim_{n \rightarrow \infty} G_{2n}(q) \cdot q^{-\frac{(2n)^2}{4}} = \frac{\vartheta_3(0, q^{-1})}{\phi(q^{-1})}.$$

Here,  $\phi(x)^{-1} = \prod_{m=1}^{\infty} (1 - x^m)^{-1}$  denotes Euler's generating function for the partition numbers, and  $\vartheta_2, \vartheta_3$  are the Jacobi theta functions

$$\vartheta_2(z, q) = \sum_{k=-\infty}^{\infty} q^{(k+\frac{1}{2})^2} e^{(2k+1)iz},$$



$$\vartheta_3(z, q) = \sum_{k=-\infty}^{\infty} q^{k^2} e^{2kiz}.$$

The limits differ by

$$(3.22) \quad g_{2\infty}(q) - g_{2\infty+1}(q) = \frac{\vartheta_4(0, q^{-\frac{1}{4}})}{\phi(q^{-1})},$$

where  $\vartheta_4$  is the Jacobi theta function

$$\vartheta_4(z, q) = \sum_{k=-\infty}^{\infty} (-1)^k q^{k^2} e^{2kiz}.$$

*Proof.* It remains to prove (3.22). This follows from the identity  $\vartheta_4(z, q) = \vartheta_3(2z, q^4) - \vartheta_2(2z, q^4)$  [27, pp. 464].  $\square$

For completeness, we take a closer look at the numbers  $g_{2\infty+1}(q)$ ,  $g_{2\infty}(q)$  and their differences.

**Corollary 3.7.** *If  $q$  is a prime power (in fact  $q \geq 2$ )*

$$(3.23) \quad g_{2\infty}(2) \geq g_{2\infty}(q) > g_{2\infty+1}(q) > 0.$$

For large values of  $q$  we have

$$(3.24) \quad g_{2\infty+1}(q) \cdot q^{\frac{(2n+1)^2}{4}} \sim 2q^{n(n-1)},$$

$$(3.25) \quad g_{2\infty}(q) \cdot q^{n^2} \sim q^{n^2},$$

and consequently

$$(3.26) \quad \lim_{q \rightarrow \infty} g_{2\infty+1}(q) = 0,$$

$$(3.27) \quad \lim_{q \rightarrow \infty} g_{2\infty}(q) = 1.$$

*Proof.* Our statements can be deduced from the Jacobi triple product identity. Namely, we have  $\vartheta_4(0, q^{-1}) = \prod_{m=1}^{\infty} (1 - q^{-2m})(1 + q^{-(2m-1)})^2$  which is  $> 0$  if  $q > 1$ . Since  $\phi(q^{-1}) > 0$  for  $q > 1$ , the strict inequality  $g_{2\infty}(q) > g_{2\infty+1}(q)$  is established. For the obviously sharp bound  $g_{2\infty}(2) \geq g_{2\infty}(q)$  we look at

$$(3.28) \quad \frac{\vartheta_3(0, q^{-1})}{\phi(q^{-1})} = \frac{\prod_{m=1}^{\infty} (1 - q^{-2m})(1 + q^{-(2m-1)})^2}{\prod_{m=1}^{\infty} (1 - q^{-m})} \\ = \prod_{m=1}^{\infty} (1 + q^{-m})(1 + q^{-(2m-1)})^2$$

which is a product of compositions of monotonic functions on  $q > 1$ . This identity also shows (3.25) and (3.27). To prove  $g_{2\infty+1}(q) > 0$ , (3.24) and (3.26) one considers

$$(3.29) \quad \frac{\vartheta_2(0, q^{-1})}{\phi(q^{-1})} = \frac{2q^{-\frac{1}{4}} \prod_{m=1}^{\infty} (1 - q^{-2m})(1 + q^{-2m})^2}{\prod_{m=1}^{\infty} (1 - q^{-m})}$$

TABLE 1. Asymptotic Galois numbers

$q$	$g_{2\infty+1}(q)$	$g_{2\infty}(q)$	$g_{2\infty}(q) - g_{2\infty+1}(q)$
2	7.371949491	7.371968801	0.0000193107
3	3.018269046	3.019783846	0.0015147993
5	1.829548122	1.845509008	0.0159608865
7	1.499386995	1.537469387	0.0380823915
11	1.229171217	1.312069129	0.0828979124
13	1.155207999	1.258137150	0.1029291515
17	1.054013475	1.191906557	0.1378930825
19	1.016940655	1.170103722	0.1531630663
23	0.9584786871	1.138621162	0.1801424752
29	0.8947912163	1.108510891	0.2137196747
$29^{2011}$	$1.203473556 \cdot 10^{-735}$	1.000000000	1.0000000000

$$= 2q^{-\frac{1}{4}} \prod_{m=1}^{\infty} (1 + q^{-m})(1 + q^{-2m})^2. \quad \square$$

For some prime powers  $q$ , the numbers  $g_{2\infty+1}(q)$ ,  $g_{2\infty}(q)$  and their differences have been listed in Table 1. The table has been produced in Mathematica with the following functions (up to a 10 digit precision:  $N[ , 10]$ ) for  $g_{2\infty+1}(q)$ ,  $g_{2\infty}(q)$  and  $g_{2\infty}(q) - g_{2\infty+1}(q)$ , respectively.

```
f[q_]:=N[EllipticTheta[2,0,1/q]1/QPochhammer[1/q,1/q],10]
g[q_]:=N[EllipticTheta[3,0,1/q]1/QPochhammer[1/q,1/q],10]
h[q_]:=N[EllipticTheta[4,0,q^(-1/4)]1/QPochhammer[1/q,1/q],10]
```

Certainly, our Theorem 3.2 allows an implementation for evaluating the asymptotic initial values of generalized Galois numbers.

**Remark 3.8.** Corollary 3.6 can be derived by the character formula of Feingold and Lepowsky [4, Theorem 4.5] for the basic representation of  $\widehat{\mathfrak{sl}}_2$ . That is,

$$(3.30) \quad \chi(V(\Lambda_0)) = \sum_{k=0}^{\infty} p(k) e^{\Lambda_0 - k\delta} \sum_{l=-\infty}^{\infty} e^{-l^2\alpha_0} e^{-l(l+1)\alpha_1},$$

where  $p(k)$  is the partition function that counts the number of ways to write  $k$  as a sum of positive integers. In fact, Kac's character formula [12, (3.37)] reduces to this expression (see [12, (3.39)]), and our proof of Theorem 3.2 reduces to this setting.

#### 4. APPLICATIONS TO LINEAR $q$ -ARY CODES

To describe the asymptotic number of non-equivalent binary  $n$ -codes in terms of the classical Galois numbers  $G_n(2)$ , Wild [28, 29] examines

numbers  $d_1(q), d_2(q)$  (see Lemma 1 in both articles) which, in the notation of Corollary 3.6, are defined as

$$\begin{aligned} d_1(q) &= g_{2\infty+1}(q), \\ d_2(q) &= g_{2\infty}(q). \end{aligned}$$

He proves that they are positive constants (depending on  $q$ ) less than 32, gives a numerical evaluation method by use of the recursion formula of Goldman and Rota [7, (5)], evaluates  $d_1(q), d_2(q)$  numerically for  $q = 2$ , and shows  $d_1(q) < d_2(q)$  for general  $q$ . Now, the detailed analytic behavior of those numbers can be extracted from Corollary 3.6 and Corollary 3.7 (see also Table 1 for examples).

For a general prime power  $q$ , Hou [9, 11] derives asymptotic equivalences for the numbers of linear  $q$ -ary codes under three notions of equivalence. That is, the permutation equivalence ( $\mathfrak{S}$ ), the monomial equivalence ( $\mathfrak{M}$ ), and semi-linear monomial equivalence ( $\Gamma$ ). He proves

$$\begin{aligned} N_{n,q}^{\mathfrak{S}} &\sim \frac{G_n(q)}{n!}, \\ N_{n,q}^{\mathfrak{M}} &\sim \frac{G_n(q)}{n!(q-1)^{n-1}}, \\ N_{n,q}^{\Gamma} &\sim \frac{G_n(q)}{n!(q-1)^{n-1}a}, \end{aligned}$$

where  $a = |\text{Aut}(\mathbf{F}_q)| = \log_p(q)$  with  $p = \text{char}(\mathbf{F}_q)$ . The asymptotic equivalence  $N_{n,2}^{\mathfrak{S}} \sim \frac{G_n(2)}{n!}$  concerns binary codes and is previously derived by Wild [28, 29]. Based on their results, the transitivity of  $\sim$  and our Corollary 3.6 produce the following list.

**Corollary 4.1.** *The asymptotic numbers of linear  $q$ -ary codes, as  $q$  is fixed and  $n \rightarrow \infty$ , under the three notions of equivalence ( $\mathfrak{S}$ ), ( $\mathfrak{M}$ ) and ( $\Gamma$ ) are given by*

$$(4.1) \quad N_{2n+1,q}^{\mathfrak{S}} \sim \frac{\vartheta_2(0, q^{-1})}{\phi(q^{-1})} \cdot \frac{q^{\frac{(2n+1)^2}{4}}}{(2n+1)!},$$

$$(4.2) \quad N_{2n,q}^{\mathfrak{S}} \sim \frac{\vartheta_3(0, q^{-1})}{\phi(q^{-1})} \cdot \frac{q^{n^2}}{(2n)!},$$

$$(4.3) \quad N_{2n+1,q}^{\mathfrak{M}} \sim \frac{\vartheta_2(0, q^{-1})}{\phi(q^{-1})} \cdot \frac{q^{\frac{(2n+1)^2}{4}}}{(2n+1)!(q-1)^{2n}},$$

$$(4.4) \quad N_{2n,q}^{\mathfrak{M}} \sim \frac{\vartheta_3(0, q^{-1})}{\phi(q^{-1})} \cdot \frac{q^{n^2}}{(2n)!(q-1)^{2n-1}},$$

$$(4.5) \quad N_{2n+1,q}^{\Gamma} \sim \frac{\vartheta_2(0, q^{-1})}{\phi(q^{-1})} \cdot \frac{q^{\frac{(2n+1)^2}{4}}}{(2n+1)!(q-1)^{2n}a},$$

$$(4.6) \quad N_{2n,q}^\Gamma \sim \frac{\vartheta_3(0, q^{-1})}{\phi(q^{-1})} \cdot \frac{q^{n^2}}{(2n)!(q-1)^{2n-1}a}.$$

Furthermore, for large prime powers  $q$  one has

$$(4.7) \quad \frac{\vartheta_2(0, q^{-1})}{\phi(q^{-1})} \cdot q^{\frac{(2n+1)^2}{4}} \sim 2q^{n(n-1)},$$

$$(4.8) \quad \frac{\vartheta_3(0, q^{-1})}{\phi(q^{-1})} \cdot q^{n^2} \sim q^{n^2}.$$

For the last two statements see Corollary 3.7.

## 5. CONCLUSION

The asymptotic enumeration method presented in this article can be summarized as follows. Once a certain specialization of Demazure characters has been identified with an interesting combinatorial function, the limit construction for affine Kac-Moody algebras can be used to carry it along towards the character of the integrable highest weight module, and derive asymptotic identities. There are at least two bottlenecks that one has to pass. First, a suitable character formula (for the limiting integrable highest weight representation) has to be available, that performs well with the chosen specialization. Fortunately, there is a great number of results and literature available, e.g. [4, 12, 14, 15] (see also [2, 5, 16]). Second, the domain, of the combinatorial function, that enumerates the objects in question must lie in the region of convergence of the limiting expressions. For example, Demazure modules specialize to tensor products of representations of the underlying finite-dimensional Lie algebra. Unfortunately, the analytic string functions limiting the tensor product multiplicities cannot be simply evaluated, for reasons of (non-)convergence, at the value 1. A much finer analysis of their asymptotic behavior when  $q \rightarrow 1$  is needed, that has to exploit the fact that we deal with modular forms [15]. Such an asymptotic analysis must take the maximal weights in the integrable highest weight module into account where those string functions emerge. Possibly borrowing and mimicking terminology from stochastic analysis like the central limit region, moderate and strong deviations region, and region of rare events. An investigation of tensor product multiplicities along those lines is planned in a future publication.

An interesting alternative project could be to re-interpret our asymptotic enumeration method geometrically through the geometric realization of Demazure and integrable highest weight modules via cohomology of Schubert and flag varieties [18].

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## REFERENCES

1. George Andrews, *The theory of partitions*, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1998, Reprint of the 1976 original.
2. Kathrin Bringmann and Ken Ono, *Some characters of Kac and Wakimoto and non-holomorphic modular functions*, Math. Ann. **345** (2009), no. 3, 547–558.
3. Roger Carter, *Lie algebras of finite and affine type*, Cambridge Studies in Advanced Mathematics, vol. 96, Cambridge University Press, Cambridge, 2005.
4. Alex Feingold and James Lepowsky, *The Weyl-Kac character formula and power series identities*, Adv. in Math. **29** (1978), no. 3, 271–309.
5. Amanda Folsom, *Kac-Wakimoto characters and universal mock theta functions*, Trans. Amer. Math. Soc. **363** (2011), no. 1, 439–455.
6. Ghislain Fourier and Peter Littelmann, *Weyl modules, Demazure modules, KR-modules, crystals, fusion products and limit constructions*, Adv. Math. **211** (2007), no. 2, 566–593.
7. Jay Goldman and Gian-Carlo Rota, *The number of subspaces of a vector space*, Recent Progress in Combinatorics (Proc. Third Waterloo Conf. on Combinatorics, 1968), Academic Press, New York, 1969, pp. 75–83.
8. Kazuhiro Hikami, *Representation of the Yangian invariant motif and the Macdonald polynomial*, J. Phys. A **30** (1997), no. 7, 2447–2456.
9. Xiang-Dong Hou, *On the asymptotic number of non-equivalent  $q$ -ary linear codes*, J. Combin. Theory Ser. A **112** (2005), no. 2, 337–346.
10. ———, *On the asymptotic number of non-equivalent binary linear codes*, Finite Fields Appl. **13** (2007), no. 2, 318–326.
11. ———, *Asymptotic numbers of non-equivalent codes in three notions of equivalence*, Linear Multilinear Algebra **57** (2009), no. 2, 111–122.
12. Victor Kac, *Infinite-dimensional algebras, Dedekind's  $\eta$ -function, classical Möbius function and the very strange formula*, Adv. in Math. **30** (1978), no. 2, 85–136.
13. ———, *Infinite-dimensional Lie algebras*, third ed., Cambridge University Press, 1990.
14. Victor Kac and Dale Peterson, *Affine Lie algebras and Hecke modular forms*, Bull. Amer. Math. Soc. (N.S.) **3** (1980), no. 3, 1057–1061.
15. ———, *Infinite-dimensional Lie algebras, theta functions and modular forms*, Adv. in Math. **53** (1984), no. 2, 125–264.
16. Victor Kac and Minoru Wakimoto, *Integrable highest weight modules over affine superalgebras and Appell's function*, Comm. Math. Phys. **215** (2001), no. 3, 631–682.
17. Shrawan Kumar, *Demazure character formula in arbitrary Kac-Moody setting*, Invent. Math. **89** (1987), no. 2, 395–423.
18. ———, *Kac-Moody groups, their flag varieties and representation theory*, Progress in Mathematics, vol. 204, Birkhäuser Boston Inc., Boston, MA, 2002.
19. Ian Macdonald, *Symmetric functions and Hall polynomials*, second ed., Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 1995, With contributions by A. Zelevinsky, Oxford Science Publications.
20. Olivier Mathieu, *Construction du groupe de Kac-Moody et applications*, C. R. Acad. Sci. Paris Sér. I Math. **306** (1988), no. 5, 227–230.

21. ———, *Formules de caractères pour les algèbres de Kac-Moody générales*, Astérisque (1988), no. 159-160, 267.
22. Leonard Rogers, *On a three-fold symmetry in the elements of Heine's series*, Proc. Lond. Math. Soc. **24** (1893), 171–179.
23. ———, *On the expansion of certain infinite products*, Proc. Lond. Math. Soc. **24** (1893), 337–352.
24. Yasmine Sanderson, *On the connection between Macdonald polynomials and Demazure characters*, J. Algebraic Combin. **11** (2000), no. 3, 269–275.
25. Gábor Szegő, *Ein Beitrag zur Theorie der Thetafunktionen*, S.B. Preuss. Akad. Wiss. Phys.-Math. KI. (1926), 242–252.
26. Ryan Vinroot, *Multivariate Rogers-Szegő polynomials and flags in finite vector spaces*, (2010), arXiv:1011.0984.
27. Edmund Whittaker and George Watson, *A course of modern analysis*, Fourth edition. Reprinted, Cambridge University Press, New York, 1962.
28. Marcel Wild, *The asymptotic number of inequivalent binary codes and nonisomorphic binary matroids*, Finite Fields Appl. **6** (2000), no. 2, 192–202.
29. ———, *The asymptotic number of binary codes and binary matroids*, SIAM J. Discrete Math. **19** (2005), no. 3, 691–699 (electronic).

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